Failure of a Stability Conjecture in General Relativity

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Abstract

By employing an exact back-reaction geometry, Helliwell-Konkowski stability conjecture is shown to fail. This happens when a test null dust is inserted to the interaction region of cross-polarized Bell-Szekeres spacetime.

I. INTRODUCTION

Some time ago Helliwell and Konkowski (HK) [1], developed a Cauchy horizon (CH) stability conjecture that uses test fields to predict the instability and type of the singularity that forms. Then a complete non-linear back reaction calculation would show that this type of singularity occurs. The CH stability conjecture due to HK is defined as follows.

Conjecture 1 *For all maximally extended spacetimes with CH, the backreaction due to a field (whose test stress-energy tensor is $T_{\mu\nu}$) will affect the horizon in one of the following manners. a)If T^{μ}_{μ}, $T_{\mu\nu}T^{\mu\nu}$ and any null dust density ρ are finite, and if the stress energy tensor T_{ab} in all parallel propagated orthonormal (PPON) frames is finite, then the CH remains non-singular. b)If T^{μ}_{μ}, $T_{\mu\nu}T^{\mu\nu}$ and any null dust density ρ are finite, but T_{ab} diverges in some PPON frames, then a non scalar curvature (NSC) singularity will be formed at the CH. c)If T^{μ}_{μ}, $T_{\mu\nu}T^{\mu\nu}$ and any null dust density ρ diverges, then an scalar curvature (SC) singularity will be formed at the CH.*

In ref.[2] (and references therein) the stability conjecture has been tested for several spacetimes. Among others the exceptional case occurs in the Bell-Szekeres (BS) [3] spacetime which describes the collision of two constant profile electromagnetic (em) waves that results in a non-singular interaction region. It was shown that, when an impacting test em field is added to one of the incoming regions of the BS spacetime, the conjecture predicts a non scalar curvature singularity (NSCS). However, the exact solution due to BS possesses a CH in the region of interaction and quasiregular singularities at the null boundaries. To our knowledge, this was a single example shown by HK to signal the failure of this conjecture.
In this paper, we further test the validity of this conjecture in the interaction region of colliding plane wave (CPW) spacetimes admitting two hypersurface non-orthogonal Killing vectors. For this purpose, we use the cross-polarized BS (CPBS) [4] solution and the outcome of the conjecture is compared with an exact back reaction solution. Our analysis verifies the failure of the conjecture in predicting the type of the singularity for this case too.
The paper is organized as follows. In section II, we review the CPBS solution and as a requirement of the conjecture we insert oppositely moving test null dust in the interaction region of CPBS spacetime. In section III, we present an exact back-reaction geometry that represents the collision of null shells (or impulsive dusts) coupled with CPBS spacetime. The paper is concluded with a conclusion in section IV.

II. TEST NULL-DUST IN THE SPACE OF CPBS SPACETIME

The metric that describes collision of em waves with the cross polarization was found to be [4]

$$ds^2 = F\left(\frac{d\tau^2}{\Delta} - \frac{d\sigma^2}{\delta}\right) - \Delta F dy^2 - \frac{\delta}{F}(dx - q\tau dy)^2 \tag{1}$$

In this representation of the metric our notations are

$$\begin{aligned} \tau &= \sin(au + bv) \\ \sigma &= \sin(au - bv) \\ \Delta &= 1 - \tau^2 \\ \delta &= 1 - \sigma^2 \\ 2F &= \sqrt{1+q^2}(1+\sigma^2) + 1 - \sigma^2 \end{aligned} \tag{2}$$

in which $0 \leq q \leq 1$ is a constant measuring the second polarization, (a, b) are constants of energy and (u, v) stand for the usual null coordinates. It can be seen easily that for $q = 0$ the metric reduces to BS. Unlike the BS metric ,however, this is conformally non-flat for $(u > 0, v > 0)$, where the conformal curvature is generated by the cross polarization. As a matter of fact this solution is a minimal extension of the BS metric. A completely different generalization of the BS solution with second polarization was given by Chandrasekhar-Xanthopoulos [5]. Their solution, however, employs an Ehlers transformation and involves two essential parameters which is therefore different from ours. Both solutions form CH in the interaction region.

Our main interest here is to test HK stability conjecture in the CPBS spacetimes. In doing this, we insert oppositely moving test null dusts in the CPBS spacetime. For simplicity we consider two different cases, the $x = constant$ and $y = constant$ projections of the spacetime. We have in the first case ($x = constant$),

$$ds^2 = \frac{e^{-M}}{2ab}\left(dt^2 - dz^2\right) - e^{-U-V} coshW\, dy^2 \tag{3}$$

where we have used the coordinates (t, z) according to

$$\begin{aligned} t &= au + bv \\ z &= au - bv \end{aligned} \tag{4}$$

The energy-momentum tensor for two oppositely moving null dusts can be chosen as

$$T_{\mu\nu} = \rho_l l_\mu l_\nu + \rho_n n_\mu n_\nu \tag{5}$$

where ρ_l and ρ_n are the finite energy densities of the dusts. The null propagation directions l_μ and n_μ are

$$\begin{aligned} l_\mu &= (a_0, 0, a_2, a_3) \\ n_\mu &= (-a_0, 0, a_2, a_3) \end{aligned}$$

with

$$a_2 = k_2 = \text{constant} \qquad\qquad a_3 = \frac{k_1}{2ab} = \text{constant}$$

$$a_0 = \frac{1}{2ab}\left(k_1^2 + \frac{2abk_2^2}{coshW}\, e^{U+V-M}\right)^{1/2}$$

We observe from (1) that

$$\frac{e^{U+V}}{coshW} = \frac{F}{\Delta F^2 + \delta q^2 \tau^2} \tag{6}$$

which is finite for $q \neq 0$. Following the requirement of the conjecture, we find the trace and scalar of the energy-momentum as

$$\begin{aligned} T^{\mu}_{\mu} &= 0 \\ T_{\mu\nu}T^{\mu\nu} &= 2\rho_l \rho_n \left(\frac{k_1^2 e^M}{ab} + \frac{2k_2^2 e^{U+V}}{coshW}\right)^2 = \text{finite} \end{aligned} \tag{7}$$

The scalar $T_{\mu\nu}T^{\mu\nu}$ reveals that as $\tau \longrightarrow 1$ it does not diverge and remains finite. Note that for a linear polarization (i.e. $q = 0$), the scalar $T_{\mu\nu}T^{\mu\nu}$ diverges as the horizon is approached indicating SCS. This particular case overlaps with the work of HK in ref. [2]. In our case, we use a non-linearly polarized metric and the outcome conflicts with the result of HK. Next we consider the energy-momentum tensor in the parallel-propagated orthonormal frame (PPON). Such frame vectors are

$$\begin{aligned} e^{\mu}_{(0)} &= \left(\sqrt{2ab}e^{M/2}, 0, 0, 0\right) \\ e^{\mu}_{(1)} &= \left(0, e^{\frac{U-V}{2}} \cosh W/2, e^{\frac{U+V}{2}} \sinh W/2, 0\right) \\ e^{\mu}_{(1)} &= \left(0, e^{\frac{U-V}{2}} \sinh W/2, e^{\frac{U+V}{2}} \cosh W/2, 0\right) \\ e^{\mu}_{(3)} &= \left(0, 0, 0, \sqrt{2ab}e^{M/2}\right) \end{aligned} \tag{8}$$

and the energy momentum tensor in this frame is given by

$$T_{(ab)} = e^{\mu}_{(a)} e^{\nu}_{(b)} T_{\mu\nu} \tag{9}$$

Then the non-zero components of $T_{(ab)}$ are;

$$\begin{aligned} T_{00} &= \left[\frac{k_2^2 e^{U+V}}{\cosh W} + \frac{k_1^2 e^M}{2ab}\right](\rho_l + \rho_n) \\ T_{01} = T_{10} &= \left\{k_2^2 (\cosh W - 1)\left[\frac{k_2^2 e^{2(U+V)}}{2\cosh W} + \frac{k_1^2 e^{U+V+M}}{4ab}\right]\right\}^{1/2} (\rho_l - \rho_n) \\ T_{02} = T_{20} &= \left\{k_2^2 (\cosh W + 1)\left[\frac{k_2^2 e^{2(U+V)}}{2\cosh W} + \frac{k_1^2 e^{U+V+M}}{4ab}\right]\right\}^{1/2} (\rho_l - \rho_n) \\ T_{03} = T_{30} &= \frac{k_1 e^M}{2ab}\left[\frac{2abk_2^2 e^{U+V-M}}{\cosh W} + k_1^2\right]^{1/2} (\rho_l - \rho_n) \\ T_{11} &= \frac{k_2^2 e^{U+V}}{2}(\cosh W - 1)(\rho_l + \rho_n) \end{aligned}$$

$$T_{13} = T_{31} = \frac{k_1 k_2}{\sqrt{2ab}} e^{(U+V+M)/2} \sinh W/2 \, (\rho_l + \rho_n)$$
$$T_{23} = T_{32} = \frac{k_1 k_2}{\sqrt{2ab}} e^{(U+V+M)/2} \cosh W/2 \, (\rho_l - \rho_n)$$
$$T_{33} = \frac{k_1^2 e^M}{2ab} \, (\rho_l + \rho_n) \tag{10}$$

Careful analysis of these components shows that some components like ($T_{01}, T_{02}, T_{11}, T_{13}, T_{23}$) diverge as $\tau \to 1$. Consequently the conjecture predicts that the CH is unstable and transforms into an NSCS.
By a similar procedure we investigate the $y = constant$ projection of the spacetime and obtain that none of the energy- momentum scalars diverges in the interaction region.

III. THE GEOMETRY OF NULL SHELLS WITH CPW.

In this section we present an exact back-reaction solution of two colliding null shells (or impulsive dusts) coupled with the CPBS spacetime. The combined metric of CPBS and colliding shells can be represented by[6]

$$ds^2 = \frac{1}{\phi^2} \, ds^2_{CPBS} \tag{11}$$

where $\phi = 1 + \alpha u \theta(u) + \beta v \theta(v)$ with (α, β) positive constants. This amounts to the substitutions

$$\begin{aligned} M &= M_0 + 2 \ln \phi \\ U &= U_0 + 2 \ln \phi \\ V &= V_0 \\ W &= W_0 \end{aligned} \tag{12}$$

where (M_0, U_0, V_0, W_0) correspond to the metric functions of the CPBS solution. Under these substitutions the scale invariant Weyl scalars remain invariant (or at most multiplied by a conformal factor) because $M - U = M_0 - U_0$ is the combination that arise in those scalars. The scalar curvature, however, which was zero in the case of CPBS now arises as nonzero. The Weyl and Maxwell scalars of the new solution are given in the Appendix.
It is clearly seen that, the scalar curvature diverges as $\tau \to 1$ (or equivalently $au+bv \to \pi/2$). It is further seen by choosing $\beta = 0$, that even a single shell gives rise to a divergent back reaction by the spacetime. The horizon, in effect, is unstable and transforms into a SCS in the presence of colliding shells or even a single propagating null shell.

IV. CONCLUSION

In this paper, we have tested the HK stability conjecture in the CPBS spacetime. Similar analysis was done by HK for the BS spacetime (linear polarization). However they were unable to compare their results with an exact back-reaction solution. We confirm their results

by taking the limit as $q \to 0$. The line element (1) reduces to the BS and the expression (6) that appears in equation (7) diverges on the horizon ($\tau = 1$) indicating an SCS. For a linearly polarized metric (which is BS) the conjecture finds correctly the nature of the singularity. But for non-colinear polarization (which is the CPBS) the conjecture predicts a NSCS. whereas the exact back-reaction solution indicates SCS. Therefore, the HK stability conjecture fails to predict the correct nature of the singularity in the non-colinear metrics. Recently, we have also shown [7] that, the inner horizon of Kerr-Newman black hole, displays a double character with respect to different perturbing potentials. In the case of null dust we have shown that the inner horizon is transformed into a spacelike SCS, however, the inclusion of particular scalar fields creates null singularities on the inner horizon of Kerr-Newman black hole. All of these outcomes are supported with exact back-reaction solutions.
Our overall impression about the conjecture is that it can be used to check the instability of the CHs, but is not reliable in determining the type of the singularity.

APPENDIX: THE WEYL AND MAXWELL SCALARS

The non-zero Weyl and Maxwell scalars for the collision of null shells in the background of CPBS spacetime are found as follows.

$$\Psi_2 = (\Psi_2)_{(CPBS)} \tag{13}$$

$$\Psi_4 = (\Psi_4)_{(CPBS)} \tag{14}$$

$$\Psi_0 = (\Psi_0)_{(CPBS)} \tag{15}$$

$$\begin{aligned} 4\phi e^{-M}\Phi_{11} = &\left[(a\beta + \alpha b)\tan(au + bv)\right. \\ &\left.+(a\beta - \alpha b)\tan(au - bv)\right]\theta(u)\theta(v) \end{aligned} \tag{16}$$

$$\begin{aligned} 4\phi e^{-M}\Lambda = &\left[(a\beta + \alpha b)\tan(au + bv) + (a\beta - \alpha b)\tan(au - bv)\right. \\ &\left.+\frac{4\alpha\beta}{\phi}\right]\theta(u)\theta(v) \end{aligned} \tag{17}$$

$$\begin{aligned} \Phi_{22} = &(\Phi_{22})_{CPBS} + \left(\frac{\alpha e^M}{\phi}\right)\left[\delta(u)\right. \\ &\left.-\theta(u)\left(a\Pi + \frac{u}{(1-u^2)(1-v^2)}\right)\right] \end{aligned} \tag{18}$$

$$\Phi_{00} = (\Phi_{00})_{CPBS} + \left(\frac{\beta e^M}{\phi}\right)\Big[\delta(v)$$

$$+\theta(v)\left(b\Pi - \frac{v}{(1-u^2)(1-v^2)}\right)\Big] \tag{19}$$

$$\Phi_{02} = (\Phi_{02})_{CPBS} + \left(\frac{e^M}{4FY\phi}\right)\left[\frac{1}{F}\left(\alpha Q\theta(u) + \beta P\theta(v)\right)\right.$$

$$\left. + iq\left(\alpha L\theta(u) + \beta K\theta(v)\right)\right] \tag{20}$$

where

$$\phi = 1 + \alpha u\theta(u) + \beta v\theta(v)$$

$$Q = b\Big[2q^2\sin(au+bv)\cos(au-bv) - F^2\left(\tan(au-bv) + \tan(au+bv)\right)$$

$$-2F\cos(au-bv)\sin(au-bv)\left(\sqrt{1+q^2}-1\right)\Big]$$

$$P = a\Big[2q^2\sin(au+bv)\cos(au-bv) + F^2\left(\tan(au-bv) - \tan(au+bv)\right)$$

$$+2F\cos(au-bv)\sin(au-bv)\left(\sqrt{1+q^2}-1\right)\Big]$$

$$Y = \left(1 + \frac{q^2}{F^2}\tan(au+bv)\sin(au+bv)\cos(au-bv)\right)^{1/2}$$

$$K = \frac{a}{\sqrt{\cos(au+bv)\cos(au-bv)}}\left[\frac{\cos(au-bv)}{\cos(au+bv)} + \sin 2au\right.$$

$$\left. - \frac{2\left(\sqrt{1+q^2}-1\right)\sin(au+bv)\cos(au-bv)\tan(au-bv)}{F}\right]$$

$$L = \frac{b}{\sqrt{\cos(au+bv)\cos(au-bv)}}\left[\frac{\cos(au-bv)}{\cos(au+bv)} + \sin 2bv\right.$$

$$\left. + \frac{2\left(\sqrt{1+q^2}-1\right)\sin(au+bv)\cos(au-bv)\tan(au-bv)}{F}\right]$$

$$\Pi = \frac{\left(\sqrt{1+q^2}-1\right)\sin(2au-2bv)}{\sqrt{1+q^2}+1+\left(\sqrt{1+q^2}-1\right)\sin^2(au-bv)}$$

and the subscript (CPBS) refers to the expressions given in ref. [4].

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